Relational Modality*

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Abstract

Saul Kripke's thesis that ordinary proper names are rigid designators is supported by widely shared intuitions about the occurrence of names in ordinary modal contexts. By those intuitions names are scopeless with respect to the modal expressions. That is, sentences in a pair like

(a) Aristotle might have been fond of dogs
(b) Concerning Aristotle, it is true that *he* might have been fond of dogs

will have the same truth value. The same does not in general hold for definite descriptions. If we, like Kripke, account for this difference by means of the *intensions* of names and descriptions, we have to conclude that names do not in general have the same intension as any normal, identifying description.

However, the difference in scope behavior between names and description can be accounted for alternatively by appeal to the semantics of the *modal expressions*. On the account we suggest, dubbed 'relational modality', simple singular terms, like proper names, contribute to modal contexts simply by their actual world reference, not by their (standard) intension. The relational modality account turns out to be fully equivalent with the rigidity account when it comes to truth of modal and non-modal sentences (with respect to the actual world), and hence supports the same basic intuitions. Given an alternative definition of consequence for relational modality, and a restriction to models with reflexive accessibility relations and non-empty world-bound domains, relational modality also turns out to be model theoretically equivalent with rigidity semantics with respect to logical consequence.

Here we introduce the semantics, give the truth definition for relational modality models, and prove the equivalence results.

Keywords: *definite descriptions, logical consequence, modality, necessity, possible worlds semantics, proper names, rigid designators, truth.*

*The present paper is part of a bigger joint project on names, general terms, and modality by the two authors. For a more philosophical discussion, see Glüer and Pagin 2006*a*. The formal work of the present paper has been done by the second author. An original hint of how to go about the technical task was given by Krister Segerberg. An earlier version of this paper, Glüer and Pagin 2006*b*, appeared in a Festschrift for Segerberg. For the clearer and more streamlined presentation in this version we are much indebted to comments by Larry Moss.

1 Introduction

In *Naming and Necessity*, Saul Kripke presented a number of soon classical arguments against the description theory of proper names. The perhaps most influential one is known as the *modal argument*. Kripke argued that proper names in general cannot have the same intensions as co-referring definite descriptions, since substituting the one for the other in modal contexts can change truth value. The intuitions on which this argument is based are widely shared and very robust. Kripke suggested to explain them by the doctrine of rigid designation.

In Glüer and Pagin 2006*a* we suggest an alternative explanation, one that is compatible with the description theory of names. We agree with Kripke that in ordinary modal thinking we operate with concepts of *de re* modality. In ordinary modal thinking, that is, we are interested in the objects we refer to, no matter how they are designated, and we want to know what would be true of these very objects in counterfactual circumstances. The intuitions made use of in Kripke's modal argument testify to this feature of ordinary modal reasoning; these are data to be accepted and explained by any good semantic theory. However, we do not agree that the best way of explaining them is by means of a thesis concerning nothing but the intension of names. The observed phenomena, we claim, essentially depend on *de re* nature of ordinary modal thinking and are, therefore, better explained in terms of a semantic interplay between names and modal expressions. We propose such an alternative semantics for ordinary modal expressions: Its basic idea is that, in ordinary modal contexts, names and other simple singular terms occur *referentially*.[1] In the companion paper we contrast our account (mainly) with Kripke's account, and elaborate on comparative merits. The aim of the present paper is to present our own account in more technical detail.

In the next section, we shall present informally the idea of relational modality and the accompanying account of proper names in modal contexts. In section 3, we provide a formal truth definition for a quantificational language with a relational necessity operator, and prove that this language is semantically equivalent with a classical (notional) modal language with rigid singular terms, with respect to truth (in the actual world) of sentences with modal operators. In section 4, we give a definition of logical consequence for relational modality and show that it is equivalent with logical consequence in standard possible worlds semantics (with rigid individual constants). The equivalence holds in the sense that the consequence relation in a class of *rigid* models $\mathcal{C}$ will be the same as the consequence relation in a certain corresponding *wider* class $\mathcal{C}'$ of models where the rigidity requirement is dropped. The result depends on the requirement that both classes of models have reflexive accessibility relations and non-empty world-bound domains. As is shown, the equivalence holds for the usual systems of modal logic.

[1] On our account, linguistic meaning cannot be equated with intension, since a difference between the semantic contributions of names and descriptions in modal contexts is induced by the semantics for modal expressions. In the companion paper, we therefore suggest to identify linguistic meaning with a pair of intensions. Cf. Glüer and Pagin 2006*a*, esp. section 5, for more on this. As we explain there in some detail, this does not make our account into a version of two-dimensionalist semantics.

2 Relational modality

Relational modality is intended to account for the natural language modal intuitions that Kripke originally appealed to, without invoking the rigidity thesis about proper names. According to these intuitions the two sentences:

(1) Aristotle might not have gone into pedagogy

(2) The teacher of Alexander might not have gone into pedagogy

are not equivalent (cf. Kripke 1980, pp. 61-63). The basic intuition is that (1) is true, while (2) is false. According to Kripke's account, the difference between (1) and (2) depends on a difference in intension between the name 'Aristotle' and and the description 'the teacher of Alexander'. This difference induces a corresponding difference in intension between the two simple sentences (3) and (4):

(3) Aristotle did not go into pedagogy

(4) The teacher of Alexander did not go into pedagogy

(cf. Kripke 1980, pp. 6f). According to Kripke, the *intensional* difference between (3) and (4) explains the extensional difference between (1) and (2). On this view it is the semantic difference between the name and the description that is decisive, not the syntactic difference. It is true that, on Kripke's view, there is an intensional property common to all the (semantically non-empty) members of the syntactic category of proper names, the property of having a constant function from worlds to objects as intension. But on Kripke's view, that is not a property that exclusively belongs to names. Every rigid definite description has it, too. The syntactic differences between names and descriptions are not, on this view, relevant to modal contexts. What is relevant is only that some descriptions are not rigid.

According to our account, the difference between (1) and (2) depends on a feature of ordinary modal thinking, not on the intensions of names (as standardly conceived; see above, note 1). When, in ordinary modal thinking, we consider alternative possibilities, we are interested in alternative scenarios involving the objects we refer to. We are interested in what might have happened to *these very objects*, regardless of how their names are evaluated with respect to those alternative scenarios. At least, this is an empirical hypothesis of the present authors about ordinary modal thinking, and hence about the ordinary modal concepts expressed by locutions like 'possibly', 'necessarily', 'it might have been' and 'it would have been', as used in everyday discourse.

In brief, the proposal is that simple singular terms, including proper names, occur referentially in the contexts of ordinary (alethic) modal expressions.[2] On the present account,

[2] In Kaplan's terminology (Kaplan 1986, p. 230), the position of a singular term within a sentence is *open to substitution* if the result of replacing a term in that position by a co-referential one does not affect the truth value of the sentence. A sentential context is then *referentially opaque*, in Quine's terminology (Quine 1952, p. 142), if any sentence (i.e. sentence occurrence) embedded in that context loses the positions open to substitution that it has on its own. In Kaplan 1986 it is argued against Quine that a position that is not open to substitution can nevertheless contain a variable that is bound by an initially placed quantifier (as

these contexts are intensional with respect to other types of expression, in particular first order predicates. Because of this, the evaluation of (1) is as follows: (1) is true if, and only if, what 'Aristotle' actually refers to did not, in some possible world, go into pedagogy. (2), on the other hand, is true if, and only if, what 'Alexander' actually refers to is such that, in some possible world, his teacher (in that world) did not go into pedagogy (in that world). Since, intuitively, on this evaluation, (1) is true and (2) is false, the semantic difference has been accounted for without the appeal to the rigidity thesis. Moreover, the syntactic difference between the name and the description plays a role, for names, like all simple singular terms, contribute to truth and falsity in modal sentences with their actual reference, regardless of their standard possible worlds intension. For all we care, 'Aristotle' might have the same intension as 'the teacher of Alexander'.

This account is not an equivalent way of making Kripke's original point. It is consistent with our account that proper names are not in general rigid. As far as we are concerned, it may be that some are and some aren't. So, assume that we believe that 'Aristotle' has the same intension as 'the teacher of Alexander'. Kripke does not believe that. On that assumption we differ with respect to the truth conditions of

(3) Aristotle did not go into pedagogy.

For Kripke, (3) is true with respect to a possible world $\boldsymbol{w}$ if, and only if, Aristotle did not go into pedagogy in $\boldsymbol{w}$. For us, given the assumption, (3) is true with respect to $\boldsymbol{w}$ if, and only if, the referent of 'the teacher of Alexander' in $\boldsymbol{w}$ did not go into pedagogy in $\boldsymbol{w}$. On our interpretation, (3) is not true in any world, whereas for Kripke it is true in some.

There is a corresponding disagreement over the metalinguistic statement

(5) (3) might have been true.

On Kripke's account, (5) is true (cf. Kripke 1980, p. 12). On our account, given the assumption, it is false. So the accounts are not equivalent. In both cases, of course, we consider what truth value (5) might have had given the meaning it actually has.

Something that does hold on our account as well as on Kripke's is the 'scopelessness' or scope indifference of names in relation to modal operators. In fact, it must hold on any account that agrees with ordinary modal intuitions. Scope indifference has sometimes been equated with the rigidity thesis, it has even been held as an alternative way of stating it (Kripke does so himself in 1980, pp. 12, fn. 15). However, this is correct only if names do *not* occur referentially in modal contexts, for then the equivalence of the wide and narrow

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in $\exists x \Box Fx$).

When we say that proper names occur referentially in modal contexts we do not mean that they occur in positions open to substitution. A name in a modal context cannot in general be replaced *salva veritate* by a description or functional expression co-referential with it. So modal contexts are opaque. However, on our interpretation, all co-referential *simple* singular terms, including proper names and free variables, can be interchanged *salva veritate* in modal contexts. This is what we mean by saying that names occur referentially in modal contexts, and that modal contexts take names transparently (we might call these contexts *semi-transparent*). Because of this feature, Kaplan's objection against Quine is exemplified by the interpretation we propose.

scope readings such as

(6) $\Box F(\text{Aristotle})$

(7) $\exists x(x = \text{Aristotle} \mathbin{\&} \Box Fx)$.

depends on the intensions of names. If names do occur referentially, then these equivalences hold as a matter of course, and whether names are rigid designators or not. In fact, this is precisely what we propose.[3]

The basic idea for effecting this, is the following interpretation of 'necessary':

(N) $\ulcorner$It is necessary that $\phi\urcorner$ is true iff ϕ is true no matter what extensions are assigned to its non-logical predicates and functional expressions.

With this clause in a truth definition, the extension of singular terms is simply left unaffected by the evaluation, while there is a variation in extension of the non-logical predicates and functional expressions. For instance,

(8) It is possible that Plato's father was richer than Aristotle's father

comes out true, on this interpretation, just if in some extension assignment to the two-place predicate '...was richer than...' and to the functional expression "...'s father" the embedded

(9) Plato's father was richer than Aristotle's father

comes out true.

Of course, this is not formally precise. As stated it is also inadequate, for there is no mention of how the extension assignment to a predicate is restricted by its meaning, nor of how assignments to different expressions may be combined. Both these problems are solved by switching to the standard framework of possible worlds semantics. The question is how to formulate the intended equivalent to (N) within that framework.

The answer comes in two very simple ideas. The first idea is what we call *actualist evaluation*. Standardly, an atomic sentence $Pt_1, ..., t_n$ is evaluated as true in a possible world w just in case the n-tuple of the referents of $t_1, ..., t_n$ in w belongs to the extension of P in w. That is, where $\mathbf{I}$ is an interpretation function assigning referents to terms and extensions to predicates in possible worlds,

(P) $\text{True}(Pt_1, \ldots, t_n, w \;\text{ iff }\; \langle \mathbf{I}(t_1, w), \ldots, \mathbf{I}(t_n, w)\rangle \in \mathbf{I}(P, w)$.

In the actualist evaluation we consider instead the referents in the *actual world*, $\boldsymbol{a}$.

(A) $\text{True}(Pt_1, \ldots, t_n, w) \;\text{ iff }\; \langle \mathbf{I}(t_1, \boldsymbol{a}), \ldots, \mathbf{I}(t_n, \boldsymbol{a})\rangle \in \mathbf{I}(P, w)$.

[3]It is also for this reason that we have chosen to call our account 'relational modality'; because of the similarity with Quine's distinction between the notional and the relational concepts of belief, in Quine 1956. This was first suggested to us by Sten Lindström.

For the predicate, the extension in w matters, but for the terms only their extension in $\boldsymbol{a}$.[4] When considering different worlds, we consider different extensions of the predicate, but just the same extension of the terms. To complete the definition of actualist evaluation one adds clauses for connectives, quantifiers and modal operators, no different from the ordinary ones (see section 3). Not surprisingly, the actualist evaluation turns out to be semantically equivalent with a standard semantics with rigid singular terms (Lemma 7). Since a rigid term denotes the same object in every world where that object exists, $\mathbf{I}(t, \boldsymbol{a})$ is bound to be the same as $\mathbf{I}(t, w)$, if t is a rigid term (and the object denoted exists in w).[5]

As stated, (A) is well defined only for simple terms, for which the reference is given primitively by the interpretation function $\mathbf{I}$. Since we prefer to stay neutral on the question whether there are complex singular terms, we have to take such terms into account. Suppose, then, that applied functional expressions, like '$g(u)$', where 'u' again is a singular term, simple or complex, are singular, and that definite descriptions, like 'the x such that Fx', or '$\imath xFx$', are singular too. In order to accommodate these terms with the desired result, (A) needs to be replaced by

(A+) $\quad$ $\text{True}(Pt_1, \ldots, t_n, w)$ iff $\langle \mathbf{V}(t_1, w), ..., \mathbf{V}(t_n, w)\rangle \in \mathbf{I}(P, w)$

where the term evaluation function $\mathbf{V}$ is defined as follows:

(V) $\quad$ $\mathbf{V}(t, w) = \mathbf{I}(t, \boldsymbol{a})$, in case t is simple
$\mathbf{V}(g(u), w) = \mathbf{I}(g, w)(\mathbf{V}(u, w))$
$\mathbf{V}(\imath xFx, w)$ = the unique object b such that $\text{True}(Fx, w)$ with b assigned to x, and undefined if there is no such object,

where $\mathbf{I}(g, w)$ is the function (in extension) assigned to g in w (for a formally more adequate statement, see section 3). By (V), simple singular terms are evaluated with respect to the actual world, while functional expressions and predicates within complex singular terms are evaluated with respect to the possible world in question.[6]

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[4]This makes it different from simply applying the actuality operator to the sentence, for that affects the evaluation of both terms and predicates: $\text{True}(A\phi, \boldsymbol{w})$ iff $\text{True}(\phi, \boldsymbol{a})$. Still, it is possible to have a semantics equivalent to the actualist evaluation by adding the actuality operator to a standard semantics with non-rigid singular terms: let each non-rigid term t occur only in the following context: 'the x such that $A(x = t)$'. The result will be that only the actual world reference of terms will matter, while predicates are evaluated as usual.

[5]An actualist evaluation semantics does not make use of the extension of a singular term in any other world than the actual world. That is, it doesn't make use of intensions of singular terms. So instead of just using an actualist evaluation as your general semantics, it would be better to drop the intensions and simply speak of the reference of a singular term. This would be to follow the Kaplan-Almog line of direct reference (see Almog 1986).

[6]It should be stressed that we have relied on the assumption that any proper name (individual constant) that has reference in some possible world also has reference in the actual world. This is philosophically well motivated. Indeed, we think that Kripke is entirely right in denying that Sherlock Holmes might have existed (Kripke 1980, pp. 157-8) since there is no actual referent with which to identify any particular individual in any particular possible world.

This goes for our account, too. It is, of course, compatible with our view that 'Sherlock Holmes' has reference in other possible worlds, but because of our interpretation of 'might have' we have to deny that Sherlock Holmes might have existed. Since actual reference is all that matters to actual truth, even of modal statements, on our interpretation, we are justified in restricting attention to names that do have actual

The second part of our proposal is to use the actualist evaluation only for the semantics of modal sentences. By contrast, if it were simply used across the board, we would have a uniform semantics for modal and non-modal sentences alike, where the actualist truth conditions of a non-modal sentence would be exactly what it contributes to the evaluation of a modal sentence containing it. Such a semantics would not have anything in particular to do with interpreting modal expressions.

The relational modality proposal, however, is to use the standard (A) for the ordinary truth conditions of atomic sentences, and to use (A+) for the semantic contribution of an atomic sentence to a modal sentence containing it. The idea, then, is to have a truth definition clause for the modal expression that runs something like this:

(M) True($\ulcorner$ It is necessary that $\phi\urcorner, w$) iff Actua-true(ϕ, w') at any world w' accessible from w,

where 'Actua-true' just means true according to the (completed) actualist evaluation. In this way we distinguish between ordinary truth conditions and the semantic properties a sentence contributes to the truth conditions of modal sentences containing it (which is its actualist truth conditions).[7]

The resulting interpretation does accommodate all our basic modal intuitions. For instance, if we adapt the proposal to 'might have', understood as 'not necessarily not', using classical predicate logic and for simplicity treating 'go into pedagogy' as a simple predicate, we get the following result for (2):

> (2) is true iff there is some accessible world w such that $\mathbf{V}$('the teacher of Alexander', w) does not belong to $\mathbf{I}$('goes into pedagogy', w). The right hand side holds iff there is a unique object b of which 'x teaches Alexander' is true (with b assigned to 'x') in w, and b does not go into pedagogy in w. Again, this holds iff there is a unique object b such that the pair of b and $\mathbf{V}$('Alexander', w) belongs to $\mathbf{I}$('x teaches y', w), and b does not go into pedagogy in w. Since $\mathbf{V}$('Alexander', w) is $\mathbf{I}$('Alexander', $\boldsymbol{a}$), i.e. Alexander, this holds iff there is a unique individual b who teaches Alexander in w, and b does not go into pedagogy in w.

This is the desired interpretation. Not only do we get the desired results for examples like (1) and (2), once a clause for the definite description construction is included in the semantics. It is also provably true that the relational modality interpretation will give the same evaluation of any formula with respect to the actual world, as the standard interpretation with rigid singular terms (see below, Theorem 1). Since truth in the actual world simply is truth, if by 'the actual world' we do mean *the actual world*, not just some entity designated as such in a model, this means that the rigidity interpretation and the relational modality interpretation

reference.

[7]This spells out a difference corresponding to Dummett's distinction between content and ingredient sense. In the semantics Kripke proposes there is no such difference, and he has been criticized by Dummett (cf. Dummett 1981, pp. 527f, and 1991, p. 48), Evans 1979, and Stanley 1997*a* as well as Stanley 1997*b* for not taking account of the distinction. In our semantics, the distinction corresponds to a real difference.

will agree with respect to simple truth and falsity of any modal and non-modal statement. Hence, they are empirically equivalent with respect to basic modal intuitions.

3 Classical and relational modality: truth

The basic idea for implementing relational modality in possible worlds semantics is to do it indirectly, via a different semantic evaluation which is actualist in the sense that for individual constants only the reference in the actual world matters for evaluation. The method is then to let the actualist evaluation kick in precisely at the clause for necessity.

The presence of a definite description operator in the formal language would complicate things considerably, and therefore we shall not include it in the following presentation. Function symbols will be included, however, and in general a functional singular term $f(t)$ may have a non-rigid interpretation even if the argument t is interpreted as rigid. Then, given a fixed first-order signature, the *first-order modal language* is the set L of formulas formed in the usual way from the signature, except that we also add $\Box\phi$ as a formula-forming operation. We often abbreviate terms $f(t_1, \ldots, t_n)$ by $f(\bar{t})$ and atomic formulas $P(t_1, \ldots, t_n)$ by $P(\bar{t})$.

A *first-order Kripke model with actual world* is a tuple

$$\mathcal{M} = \langle W, R, \mathbf{a}, Dom, \mathbf{D}, \mathbf{I} \rangle,$$

where W is a set of *worlds*, $R \subseteq W \times W$ is an *accessibility relation*, $\mathbf{a} \in W$ is the *actual world*, Dom is a set of *entities*, $\mathbf{D} : W \to \mathcal{P}(Dom)$ tells which entities exist in which worlds, and $\mathbf{I}$ is an interpretation of L in Dom in the usual way, except that the interpretation of constants and function symbols are now taken to be partial, and in addition may vary with the world (that is, the interpretation function $\mathbf{I}$ takes a world as one of its arguments). The interpretations of the predicate symbols also are allowed to vary with the world. So we shall have, for example, $\mathbf{I} : L\text{-}constants \times W \to Dom$. We shall call these structures *models* for short.

Definition 1. *A model* $\mathcal{M}$

a) *is* rigid *iff it holds for any individual constant* $c \in L$ *and any world* $w \in W$ *that*

   i) $\mathbf{I}(c, w)$ *is defined iff* $\mathbf{I}(c, \mathbf{a})$ *is defined and* $\mathbf{I}(c, \mathbf{a}) \in w$*, and*

   ii) *if* $\mathbf{I}(c, w)$ *is defined, then* $\mathbf{I}(c, w) = \mathbf{I}(c, \mathbf{a})$

b) *satisfies the* extension postulate *iff it holds for any n-ary predicate symbol* $P \in L$ *and any world* $w \in W$ *that* $\mathbf{I}(P, w) \subseteq (\mathbf{D}(w))^n$ *and any n-ary functional symbol* $f \in L$ *and any world* $w \in W$ *that* $\mathbf{I}(f, w) \subseteq (\mathcal{D}(w))^{n+1}$*,*

c) *is* classical *iff* $\mathcal{M}$ *is both rigid and satisfies the extension postulate.*

We define *three* semantics of the first-order modal language. These relate formulas ϕ to tuples $\langle \mathcal{M}, \mu, w \rangle$, where $\mathcal{M}$ is a model as above, μ : *variables* $\rightarrow$ *Dom*, and $w \in W$. So we shall be writing

$$\langle \mathcal{M}, \mu, w \rangle \models^{c} \phi \qquad \langle \mathcal{M}, \mu, w \rangle \models^{a} \phi \qquad \langle \mathcal{M}, \mu, w \rangle \models^{r} \phi$$

where $\mathcal{M}$ is a model, μ : variables $\rightarrow$ *Dom*, and $w \in W$. We next have an inductive interpretation I_μ of terms at worlds using an interpretation structure $\mathbf{I}$ from a model and an assignment μ:

(10)
$$\begin{aligned} \mathbf{I}_\mu(x, w) &= \mu(x) \\ \mathbf{I}_\mu(c, w) &= \mathbf{I}(c, w) \\ \mathbf{I}_\mu(f(t_1, \ldots, t_n), w) &= \mathbf{I}(f, w)(\mathbf{I}_\mu(t_1, w), \ldots, \mathbf{I}_\mu(t_n, w)) \end{aligned}$$

Since I is partial, I_μ is of course partial, too.

The three definitions employ three parallel *definedness relations* between formulas and terms on the one hand, and tuples $\langle \mathcal{M}, \mu, w \rangle$ on the other. The presence of function symbols requires for the actualist evaluation an interpretation function $\mathbf{V}_\mu$:

(**V**)
$$\begin{aligned} &\mathbf{V}_\mu(c, \boldsymbol{w}) = \mathbf{I}(c, \boldsymbol{a}), \text{ for an individual constant } c \\ &\mathbf{V}_\mu(x, w) = \mu(x), \text{ for a variable } x \\ &\mathbf{V}_\mu(f(t), w) = \mathbf{I}(f, w)(\mathbf{V}(t, w)) \end{aligned}$$

The definedness relations are given in Figure 1. In the step for quantified formulas, $\mu \sim_x \mu'$ means that the two valuations agree on all variables except possibly x. In the steps for function symbols and predicate symbols, the index variable i is implicitly quantified. $dom(\mathbf{I}(f, w))$ is the domain of the extension of f in w according to $\mathbf{I}$.

Definition 2. *Two models,* $\mathcal{M} = \langle \boldsymbol{W}, R, \boldsymbol{a}, Dom, \mathbf{D}, \mathbf{I} \rangle$ *and* $\mathcal{M}' = \langle \boldsymbol{W}', R', \boldsymbol{a}', Dom', \mathbf{D}', \mathbf{I}' \rangle$, are *equi-actual,* $\mathcal{M} \sim \mathcal{M}'$, *iff*

$$\boldsymbol{W} = \boldsymbol{W}', R = R', \mathbf{a} = \mathbf{a}', Dom = Dom', \mathbf{D} = \mathbf{D}'$$

and for all predicate letters P, function symbols f, individual constants c, and all worlds $\boldsymbol{w} \in \boldsymbol{W}$,

i) $\mathbf{I}(P, w) = \mathbf{I}'(P, w)$, *and*

ii) $\mathbf{I}(f, w) = \mathbf{I}'(f, w)$, *and*

iii) $\mathbf{I}(c, \boldsymbol{a}) = \mathbf{I}'(c, \boldsymbol{a}')$.

Lemma 1. *Suppose that* $\mathcal{M}$ *is classical and that* $\mathcal{M} \sim \mathcal{M}'$. *Then for all terms* t,

$$\text{Def}_c(\langle \mathcal{M}, \mu, w \rangle, t) \quad \textit{iff} \quad \text{Def}_a(\langle \mathcal{M}', \mu, w \rangle, t).$$

	$Def_c(\langle \mathcal{M}, \mu, w\rangle, -)$	$Def_a(\langle \mathcal{M}, \mu, w\rangle, -)$	$Def_r(\langle \mathcal{M}, \mu, w\rangle, -)$
x	$\mu(x) \in \mathbf{D}(w)$	$\mu(x) \in \mathbf{D}(w)$	$\mu(x) \in \mathbf{D}(w)$
c	$\mathbf{I}(c, w) \in \mathbf{D}(w)$	$\mathbf{I}(c, \mathbf{a}) \in \mathbf{D}(w)$	$\mathbf{I}(c, w) \in \mathbf{D}(w)$
$f(\bar{t})$	$\mathbf{I}_\mu(t_i, w) \in dom(\mathbf{I}(f, w))$	$\mathbf{V}_\mu(t_i, w) \in dom(\mathbf{I}(f, w))$	$\mathbf{I}_\mu(t_i, w) \in dom(\mathbf{I}(f, w))$
$t = u$	$Def_c(\langle \mathcal{M}, \mu, w\rangle, t)$ and $Def_c(\langle \mathcal{M}, \mu, w\rangle, u)$	$Def_a(\langle \mathcal{M}, \mu, w\rangle, t)$ and $Def_a(\langle \mathcal{M}, \mu, w\rangle, u)$	$Def_r(\langle \mathcal{M}, \mu, w\rangle, t)$ and $Def_r(\langle \mathcal{M}, \mu, w\rangle, u)$
$P(\bar{t})$	$Def_c(\langle \mathcal{M}, \mu, w\rangle, t_i)$	$Def_a(\langle \mathcal{M}, \mu, w\rangle, t_i)$	$Def_r(\langle \mathcal{M}, \mu, w\rangle, t_i)$
$\neg\phi$	$Def_c(\langle \mathcal{M}, \mu, w\rangle, \phi)$	$Def_a(\langle \mathcal{M}, \mu, w\rangle, \phi)$	$Def_r(\langle \mathcal{M}, \mu, w\rangle, \phi)$
ϕ & ψ	$Def_c(\langle \mathcal{M}, \mu, w\rangle, \phi)$ and $Def_c(\langle \mathcal{M}, \mu, w\rangle, \psi)$	$Def_a(\langle \mathcal{M}, \mu, w\rangle, \phi)$ and $Def_a(\langle \mathcal{M}, \mu, w\rangle, \psi)$	$Def_r(\langle \mathcal{M}, \mu, w\rangle, \phi)$ and $Def_r(\langle \mathcal{M}, \mu, w\rangle, \psi)$
$\forall x\phi$	$Def_c(\langle \mathcal{M}, \mu', w\rangle, \phi)$ for all $\mu \sim_x \mu'$ with $\mu'(x) \in \mathbf{D}(w)$	$Def_a(\langle \mathcal{M}, \mu', w\rangle, \phi)$ for all $\mu \sim_x \mu'$ with $\mu'(x) \in \mathbf{D}(w)$	$Def_r(\langle \mathcal{M}, \mu', w\rangle, \phi)$ for all $\mu \sim_x \mu'$ with $\mu'(x) \in \mathbf{D}(w)$
$\Box\phi$	$Def_c(\langle \mathcal{M}, \mu, w\rangle, \phi)$ for all wRw'	$Def_a(\langle \mathcal{M}, \mu, w\rangle, \phi)$ for all wRw'	$Def_a(\langle \mathcal{M}, \mu, w\rangle, \phi)$ for all wRw'

Figure 1: The definedness relations between terms and formulas, and triples $\langle \mathcal{M}, \mu, w\rangle$.

Proof. By induction on t. The base case for constants uses rigidity, and the other steps are easy. *QED*

Lemma 2. *Suppose that $\mathcal{M}$ is classical and that $\mathcal{M} \sim \mathcal{M}'$. Then for all formulas ϕ,*

$$\mathrm{Def}_c(\langle \mathcal{M}, \mu, w\rangle, \phi) \quad \textit{iff} \quad \mathrm{Def}_a(\langle \mathcal{M}', \mu, w\rangle, \phi)$$

Proof. By induction on ϕ. The base case for atomic formulas uses Lemma 1. *QED*

Lemma 3. *Suppose that $\mathcal{M}$ is classical and that $\mathcal{M} \sim \mathcal{M}'$. Then for all terms t,*

$$\mathrm{Def}_c(\langle \mathcal{M}, \mu, \mathbf{a}\rangle, t) \quad \textit{iff} \quad \mathrm{Def}_r(\langle \mathcal{M}', \mu, \mathbf{a}\rangle, t).$$

Proof. Immediate from Definition 2. *QED*

Lemma 4. *Suppose that $\mathcal{M}$ is classical and that $\mathcal{M} \sim \mathcal{M}'$. Then for all formulas ϕ,*

$$\mathrm{Def}_c(\langle \mathcal{M}, \mu, \mathbf{a}\rangle, \phi) \quad \textit{iff} \quad \mathrm{Def}_r(\langle \mathcal{M}', \mu, \mathbf{a}\rangle, \phi)$$

Proof. By induction on ϕ. The base case for atomic formulas uses Lemma 3. The modal step uses Lemma 2. *QED*

Using the definedness relations in Figure 1 and the semantics of terms from (10), we

	$Tr_c(\langle\mathcal{M},\mu,w\rangle,-)$	$Tr_a(\langle\mathcal{M},\mu,w\rangle,-)$	$Tr_r(\langle\mathcal{M},\mu,w\rangle,-)$
$t=u$	$\mathbf{I}_\mu(t,w)=\mathbf{I}_\mu(u,w)$	$\mathbf{V}_\mu(t,w)=\mathbf{V}_\mu(u,w)$	$\mathbf{I}_\mu(t,w)=\mathbf{I}_\mu(u,w)$
$P(\bar{t})$	$\langle\mathbf{I}_\mu(\bar{t},w)\rangle\in\mathbf{I}(P,w)$	$\langle\mathbf{V}_\mu(\bar{t},w)\rangle\in\mathbf{I}(P,w)$	$\langle\mathbf{I}_\mu(\bar{t},w)\rangle\in\mathbf{I}(P,w)$
$\neg\phi$	$Def_c(\langle\mathcal{M},\mu,w\rangle,\phi)$ but$\langle\mathcal{M},\mu,w\rangle\not\models^{c}\phi$	$Def_a(\langle\mathcal{M},\mu,w\rangle,\phi)$ but$\langle\mathcal{M},\mu,w\rangle\not\models^{a}\phi$	$Def_r(\langle\mathcal{M},\mu,w\rangle,\phi)$ but$\langle\mathcal{M},\mu,w\rangle\not\models^{r}\phi$
$\phi\ \&\ \psi$	$\langle\mathcal{M},\mu,w\rangle\models^{c}\phi$ and $\langle\mathcal{M},\mu,w\rangle\models^{c}\psi$	$\langle\mathcal{M},\mu,w\rangle\models^{a}\phi$ and $\langle\mathcal{M},\mu,w\rangle\models^{a}\psi$	$\langle\mathcal{M},\mu,w\rangle\models^{r}\phi$ and $\langle\mathcal{M},\mu,w\rangle\models^{r}\psi$
$\forall x\phi$	$\langle\mathcal{M},\mu,w\rangle\models^{c}\phi$ for all $\mu\sim_x\mu'$ with $\mu'(x)\in\mathbf{D}(w)$	$\langle\mathcal{M},\mu,w\rangle\models^{a}\phi$ for all $\mu\sim_x\mu'$ with $\mu'(x)\in\mathbf{D}(w)$	$\langle\mathcal{M},\mu,w\rangle\models^{r}\phi$ for all $\mu\sim_x\mu'$ with $\mu'(x)\in\mathbf{D}(w)$
$\Box\phi$	$\langle\mathcal{M},\mu,w'\rangle\models^{c}\phi$ whenever wRw'	$\langle\mathcal{M},\mu,w'\rangle\models^{a}\phi$ whenever wRw'	$\langle\mathcal{M},\mu,w'\rangle\models^{r}\phi$ whenever wRw'

Figure 2: The three semantics

now define the three truth relations in Figure 2. We write $\langle\mathbf{I}_\mu(\bar{t},w)\rangle$ as an abbreviation of $\langle\mathbf{I}_\mu(t_1,w),\ldots,\mathbf{I}_\mu(t_n,w)\rangle$. Similarly for $\mathbf{V}_\mu$.

On the basis of these definitions and the given lemmas we can now relate the semantics with respect to truth. We first observe that

Lemma 5. *For any classical model $\mathcal{M}$, if $\langle\mathcal{M},\mu,w\rangle\models^{c}\phi$, then* $\mathrm{Def}_c(\langle\mathcal{M},\mu,w\rangle,\phi)$, *and if $\mathcal{M}'\sim\mathcal{M}$ and $\langle\mathcal{M}',\mu,w\rangle\models^{r}\phi$, then* $\mathrm{Def}_r(\langle\mathcal{M}',\mu,w\rangle,\phi)$.

Proof. By induction on ϕ. *QED*

Lemma 6. *Suppose that $\mathcal{M}$ is classical and that $\mathcal{M}\sim\mathcal{M}'$. Then, with $\mathbf{I}_\mu$ for $\mathcal{M}$ and $\mathbf{V}_\mu$ for $\mathcal{M}'$, for all terms t and worlds $w\in W$,*

$$\mathbf{I}_\mu(t,w)=\mathbf{V}_\mu(t,w)$$

Proof. By induction on t. The atomic case uses rigidity. *QED*

Lemma 7. *Suppose that $\mathcal{M}$ is classical and that $\mathcal{M}\sim\mathcal{M}'$. Then for all formulas ϕ and worlds $w\in W$,*

$$\langle\mathcal{M},\mu,w\rangle\models^{c}\phi \quad \textit{iff} \quad \langle\mathcal{M}',\mu,w\rangle\models^{a}\phi$$

Proof. By induction on ϕ. For an atomic formula $t=u$, note that by Lemma 6, $\mathbf{I}_\mu(t,w)=\mathbf{I}_\mu(u,w)$ if and only if $\mathbf{V}_\mu(t,w)=\mathbf{V}_\mu(u,w)$. For atomic formulas $P(\bar{t})$, the reason is the same. The induction step for $\neg\phi$ uses Lemma 2. The other induction steps are straightforward. *QED*

We can now prove the equivalence of classical and relational models with respect to truth (in the actual world).

Theorem 1. *Suppose that $\mathcal{M}$ is classical and that $\mathcal{M} \sim \mathcal{M}'$. Then for all formulas ϕ,*

$$\langle \mathcal{M}, \mu, \mathbf{a} \rangle \models^{c} \phi \quad \textit{iff} \quad \langle \mathcal{M}', \mu, \mathbf{a} \rangle \models^{r} \phi$$

Proof. By induction on ϕ. The atomic cases are obvious. The induction step for $\neg\phi$ uses Lemma 2. The induction step for $\Box\phi$ uses Lemma 7. *QED*

4 Classical and relational modality: consequence

Let's turn now to the issues of consequence and validity. As usual, validity is just the special case of consequence where the set of premises is empty. It will turn out that there is concept of consequence for relational modality that is almost equivalent with the standard concept of consequence for classical possible worlds semantics. Where the standard concept has it that ϕ is a consequence of Γ iff ϕ is true at a world w if Γ is true at w, for all worlds in all models, the relational counterpart is that ϕ is a consequence of Γ iff ϕ is true at the actual world $\mathbf{a}$ if Γ is true at $\mathbf{a}$, in all models. We shall call the standard notion *universal consequence* and the relational counterpart *actualist consequence*.[8]

These two notions are almost but not completely equivalent. Because of denotation failures, they diverge both with respect to quantification and with respect to modality. Consider the pair of sentences

(11) a) $\forall x(Ptx)$
b) Ptt

where Ptt is atomic and t an individual constant. With the requirement that any term that does have an interpretation in a model has a denotation in the actual world in that model, whenever (11a) is true at the actual world in some model, the term t has a denotation in the actual world, and then if (11a) is true in the actual world, so is (11b). So, it is a relational consequence.

However, it is not a classical consequence. For consider a world w with an empty domain. The premise (11a) is vacuously true with respect to w, and the sentence also is vacuously well-defined with respect to that world and the relevant assignment μ. For this holds if Ptx is well-defined with respect to that world for all x-variants μ' of μ such that $\mu'(x) \in \mathbf{D}(w)$, and since $\mathbf{D}(w) = \varnothing$, this condition is vacuously fulfilled by all assignments. But (11b) is not well-defined with respect to w, since t obviously has no denotation there. So, (11b) is not true at w, and the consequence fails.

Similarly, consider the pair of sentences

(12) a) $\Box Pt$

[8] The validity version of this notion was called *real world validity* in Davies and Humberstone 1980.

b) $\neg(Pt\,\&\,\neg Pt)$

where Pt is atomic and t an individual constant. With the requirement that any term that does have an interpretation in a model has a denotation in the actual world in that model, whenever (12a) is true at the actual world in some model, the term t has a denotation in the actual world, and then (12b), being a tautology, is true at the actual world, since it is true at any world where the term does have a denotation. So, (12b) is a relational consequence of (12a).

Again, however, it is not a classical consequence. For consider a world w_k in a classical model $\mathcal{M}$ such that (12a) is true at w_k in $\mathcal{M}$. This requires exactly that Pt is true at all worlds w accessible from w_k, and hence that t has a denotation in all of these worlds. Hence, in all those worlds, (12b) is true as well. However, it is not required that w_k be accessible from itself, and so t need not have a denotation in w_k. If not, (12b) will be neither true nor false in w_k. So the consequence fails.

Nevertheless, the notions *are* equivalent if it is required both that world-bound domains be non-empty and that worlds be accessible from themselves. That is, the notions are equivalent with respect to reflexive models with non-empty world-bound domains (and with the rigidity requirement dropped in the relational case). Model theoretically, this is a limitation. Philosophically, the limitation is not severe, since both (11) and (12) are pairs where the the second member intuitively follows from the first. Moreover, we want to infer truth from necessity, i.e. p from *necessarily* p, and the validity of this inference is guaranteed, from a possible worlds semantics perspective, only when the accessibility relation is reflexive.

It may be noted that in case we assume an invariant non-empty world-bound domain of individuals, the counter-examples do not go through. And as can easily be seen by inspecting the proof of Lemma 9, with this assumption there is equivalence between classical and relational consequence, without the reflexivity restriction. However, since this assumption makes the comparison less interesting, we shall proceed without it.

We shall now establish the equivalence with respect to non-empty reflexive models. For universal consequence, if $\mathcal{C}$ is a class of models, we write $\Gamma \overset{c}{\models}_{\mathcal{U}} \phi$ *on* $\mathcal{C}$ if whenever $\mathcal{M}$ is a model in $\mathcal{C}$, $w \in W$, and μ an assignment, if $\langle \mathcal{M}, \mu, w\rangle \overset{c}{\models} \psi$ for each $\psi \in \Gamma$, then also $\langle \mathcal{M}, \mu, w\rangle \overset{c}{\models} \phi$. This is a standard notion of consequence from modal logic. Analogously, we define $\Gamma \overset{a}{\models}_{\mathcal{U}} \phi$ *on* $\mathcal{C}$ and $\Gamma \overset{r}{\models}_{\mathcal{U}} \phi$ *on* $\mathcal{C}$ in the obvious ways.

For actualist consequence, we write $\Gamma \overset{a}{\models}_{\mathcal{A}} \phi$ *on* $\mathcal{C}$ if whenever $\mathcal{M}$ is a model in $\mathcal{C}$, and μ an assignment, if $\langle \mathcal{M}, \mu, \mathbf{a}\rangle \overset{a}{\models} \psi$ for each $\psi \in \Gamma$, then also $\langle \mathcal{M}, \mu, \mathbf{a}\rangle \overset{a}{\models} \phi$. We could also study versions of this for $\overset{c}{\models}$ and $\overset{r}{\models}$, but that falls outside the scope of the present paper.

We shall be concerned with two classes of models. Let $\mathcal{C}$ be the class of models $\mathcal{M}$ with the following properties: $\mathcal{M}$ is classical, and in addition R is reflexive on W and each $\mathbf{D}(w)$ is a non-empty subset of *Dom*.

Let $\mathcal{C}'$ be the class of models such that for each $\mathcal{M}$ in $\mathcal{C}'$ there is a model $\mathcal{M}'$ in $\mathcal{C}$ such that $\mathcal{M} \sim \mathcal{M}'$. Clearly, $\mathcal{C} \subset \mathcal{C}'$. Moreover, it is immediate from Definition 2 that every model $\mathcal{M}$ in $\mathcal{C}'$ has a reflexive accessibility relation R and non-empty world domains; also, it is immediate from Definitions 1 and 2 that $\mathcal{M}$ satisfies the extension postulate.

We aim to show that for any ϕ and Γ of L,

$$\Gamma \models^{c}_{\mathcal{U}} \phi \text{ on } \mathcal{C} \quad \text{iff} \quad \Gamma \models^{r}_{\mathcal{A}} \phi \text{ on } \mathcal{C}'$$

We shall do this by way of first showing that universal consequence for classical models reduces to actualist consequence:

$$\Gamma \models^{c}_{\mathcal{U}} \phi \text{ on } \mathcal{C} \quad \text{iff} \quad \Gamma \models^{c}_{\mathcal{A}} \phi \text{ on } \mathcal{C}$$

From then on, the theorem follows by the use of Theorem 1. The left-to-right part of the above biconditional is immediate, and the right-to-left part would be easy if we could vary the world to play the role of the actual world among the models of $\mathcal{C}$: whenever there is some world w in some model $\mathcal{M}$ that provides a counterexample to the universal consequence, find the model $\mathcal{M}'$ that is like $\mathcal{M}$ except for using w as its actual world, and we have a counterexample to the actualist consequence.[9] This simple strategy fails, however, because of the requirement that any individual constant that has denotation at all has denotation in the actual world. If we allow domains to vary between worlds, this requirement would not always be met when selecting some new world w as actual. Hence, we need a different idea.

The alternative idea is to find a model where the actual world has the required domain and accessibility properties. For this purpose we first define:

Definition 3. $Den(\mathbf{I}, \phi, w)$ iff for every constant c in ϕ, $\mathbf{I}(c, w)$ is defined.

Lemma 8. *For any $\phi \in L$ and $\mathcal{M} \in \mathcal{C}$, if $Def_c(\langle \mathcal{M}, \mu, w \rangle, \phi)$, then $Den(\mathbf{I}_{\mathcal{M}}, \phi, w)$.*

Proof. By induction on ϕ. The case of $\forall x \phi$ makes use of the restriction to non-empty world-bound domains. The case of $\Box \phi$ makes use of the restriction to reflexive accessibility relations. *QED*

This will be used in the proof of the following lemma:

Lemma 9. *For any $\mathcal{M} = \langle W, R, \mathbf{a}, \text{Dom}, \mathbf{D}, \mathbf{I} \rangle$ in $\mathcal{C}$ and any world w in it, there is a model with the same sets of worlds and entities*

$$\mathcal{M}' \quad = \quad \langle W, R', \mathbf{a}, \text{Dom}, \mathbf{D}', \mathbf{I}' \rangle,$$

such that

1. *$\langle W, R \rangle \cong \langle W, R' \rangle$.*
2. *For all ϕ and μ: $\langle \mathcal{M}, \mu, w \rangle \models^{c} \phi$ iff $\langle \mathcal{M}', \mu, \mathbf{a} \rangle \models^{c} \phi$.*

Proof. Let $\mathbf{a}' = w$. Let $h : W \to W$ be given by $h(\mathbf{a}) = \mathbf{a}'$, $h(\mathbf{a}') = \mathbf{a}$, and otherwise h is the identity. This h is a bijection, and its inverse h^{-1} is h itself. We define $\mathcal{M}'$ by means

[9] This would no longer hold if we were to add an actuality *operator* to the language.

of h:

i) $\mathbf{D}_{\mathcal{M}'}(w) = \mathbf{D}_{\mathcal{M}}(h(w)$, for any $w \in W$

ii) $R_{\mathcal{M}'}(w, w')$ iff $R_{\mathcal{M}}(h(w), h(w'))$, for any $w \in W$

iii) $\mathbf{I}_{\mathcal{M}'}(P, w) = \mathbf{I}_{\mathcal{M}}(P, h(w))$, for any predicate letter P in L and any $w \in W$

iv) $\mathbf{I}_{\mathcal{M}'}(f, w) = \mathbf{I}_{\mathcal{M}}(f, h(w))$, for any function symbol f in L and any $w \in W$

v) $\mathbf{I}_{\mathcal{M}'}(c, w) = \mathbf{I}_{\mathcal{M}}(c, h(w))$, for any individual constant c in L and any $w \in W$, except that $\mathbf{I}_{\mathcal{M}'}(c, w)$ is defined only if $\mathbf{I}_{\mathcal{M}}(c, \mathbf{a}')$ is defined.

It is easily verified that $\mathcal{M}'$ is a model in $\mathcal{C}$. The first part of the Lemma is immediate from the definition of $\mathcal{M}'$. Next, we show by induction on t (for the atomic case) and ϕ that

(*) For any $\phi \in L$ such that $Den(\mathbf{I}_{\mathcal{M}}, \phi, \mathbf{a}')$, it holds for any $w \in W$ that

$$Def_c(\langle \mathcal{M}', \mu, w\rangle, \phi) \quad \text{iff} \quad Def_c(\langle \mathcal{M}, \mu, h(w)\rangle, \phi).$$

We then show by induction on ϕ that

(#) For any $\phi \in L$ such that $Den(\mathbf{I}_{\mathcal{M}}, \phi, \mathbf{a}')$, it holds for any $w \in \mathbf{W}$ and that

$$\langle \mathcal{M}', \mu, w\rangle \models^{c} \phi \quad \text{iff} \quad \langle \mathcal{M}, \mu, h(w)\rangle \models^{c} \phi.$$

The negation case uses (*) above.

As a corollary of (#), we have the instance

(##) If $Den(\mathbf{I}_{\mathcal{M}}, \phi, \mathbf{a}')$, then $\langle \mathcal{M}, \mu, \mathbf{a}'\rangle \models^{c} \phi$ iff $\langle \mathcal{M}', \mu, \mathbf{a}\rangle \models^{c} \phi$.

Now, assume $\langle \mathcal{M}, \mu, \mathbf{a}'\rangle \models^{c} \phi$. By Lemma 5, $Def_c(\langle \mathcal{M}, \mu, \mathbf{a}'\rangle, \phi)$. Then by Lemma 8, $Den(\mathbf{I}_{\mathcal{M}}, \phi, \mathbf{a}')$. Hence, by (##), $\langle \mathcal{M}', \mu, \mathbf{a}\rangle \models^{c} \phi$.

Conversely, assume $\langle \mathcal{M}', \mu, \mathbf{a}\rangle \models^{c} \phi$. Again, by Lemma 5, $Def_c(\langle \mathcal{M}', \mu, \boldsymbol{a}\rangle, \phi)$. And by Lemma 8, $Den(\mathbf{I}_{\mathcal{M}'}, \phi, \boldsymbol{a})$. By the definition of $\mathcal{M}'$ and by Definition 3, this holds iff $Den(\mathbf{I}_{\mathcal{M}}, \phi, \mathbf{a}')$. Hence, by (##) again, $\langle \mathcal{M}, \mu, \mathbf{a}'\rangle \models^{c} \phi$. Thus,

(!) $\langle \mathcal{M}, \mu, \mathbf{a}'\rangle \models^{c} \phi$ iff $\langle \mathcal{M}', \mu, \mathbf{a}\rangle \models^{c} \phi$

Since $\mathcal{M}$ and $\mathbf{a}'$ were arbitrarily chosen, part 2 of the Lemma follows by generalization.

QED

With the help of Lemma 9, we can go on to prove

Lemma 10. $\Gamma \models^{c}_{\mathbf{U}} \phi$ *on* $\mathcal{C}$ *iff* $\Gamma \models^{c}_{\mathbf{A}} \phi$ *on* $\mathcal{C}$.

Proof. Left-to-right by instantiation. For right-to-left, assume that $\Gamma \models^{a}_{\mathcal{A}} \phi$ on $\mathcal{C}$, and take a tuple $\langle \mathcal{M}, \mu, w\rangle$ satisfying each $\psi \in \Gamma$. We must check that $\langle \mathcal{M}, \mu, w\rangle \models \phi$. Consider $\mathcal{M}'$ from Lemma 9. Then for all $\psi \in \Gamma$, $\langle \mathcal{M}', \mu, \mathbf{a}\rangle \models \psi$. By hypothesis, $\langle \mathcal{M}', \mu, \mathbf{a}\rangle \models \phi$ as well. By Lemma 9 again, $\langle \mathcal{M}, \mu, w \rangle \models \phi$ as desired. *QED*

From Theorem 1 we can derive

Lemma 11. *For any* $\Gamma, \phi \in L$, $\Gamma \models^{c}_{\mathcal{A}} \phi$ *on* $\mathcal{C}$ *iff* $\Gamma \models^{r}_{\mathcal{A}} \phi$ *on* $\mathcal{C}'$.

Proof. Straightforward use of Theorem 1. *QED*

Now we can conclude

Theorem 2. *For any* $\Gamma, \phi \in L$, $\Gamma \models^{c}_{\mathcal{U}} \phi$ *on* $\mathcal{C}$ *iff* $\Gamma \models^{r}_{\mathcal{A}} \phi$ *on* $\mathcal{C}'$.

Proof. Immediate from Lemmas 10 and 11. *QED*

This is the main result. It is, however, also of interest to know whether the equivalence remains under various restrictions on the accessibility relation. For instance, if we restrict our classes $\mathcal{C}$ and $\mathcal{C}'$ by requiring that R be transitive, and hence that the S4 axiom of modal logic be valid, will the equivalence still hold? The answer is yes, and is an immediate consequence of part 1 of Lemma 9 together with the main results:

Corollary 3. *For any structural restriction* ρ *on accessibility relations* R*, let* $\mathcal{C}_\rho$ *and* $\mathcal{C}'_\rho$*, respectively be the subclasses of* $\mathcal{C}$ *and* $\mathcal{C}'$ *whose accessibility relation obey* ρ*. Then for any* $\Gamma, \phi \in L$, $\Gamma \models^{c}_{\mathcal{U}} \phi$ *on* $\mathcal{C}_\rho$ *iff* $\Gamma \models^{r}_{\mathcal{A}} \phi$ *on* $\mathcal{C}'_\rho$.

Proof. No assumption except reflexivity has been made about the accessibility relation in this section. By Lemma 9.1, the $\langle W, R' \rangle$ of the model $\mathcal{M}'$ is isomorphic to $\langle W, R \rangle$ of $\mathcal{M}$, and hence any structural property, such as transitivity or symmetry, that holds of R on W also holds of R' on W. The other theorems go through as before. *QED*

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